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Projection method for Fractional Lavrentiev Regularisation method in Hilbert scales

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Abstract
We study finite dimensional Fractional Lavrentiev Regularization (FLR) method for linear ill-posed operator equations in the Hilbert scales. We obtain an optimal order error estimate under Hölder type source condition and under a parameter choice strategy. Numerical experiments confirming the theoretical results are also given.

Keywords Lavrentiev Regularization · Ill-posed problem · Hilbert scales · Discrepancy principle · Finite dimension

Mathematics Subject Classification 47A52 · 65R10 · 65J10 · 47H09 · 49J30

1 Introduction

Consider the ill-posed equation

$$Ax = y, \tag{1}$$

where $A : X \longrightarrow Y$ is a bounded linear operator between the Hilbert spaces X and Y. These type of equations have many applications for example, neural network [10], image deblurring [20], magnetic resonance [27] and others. In practice, instead of y we have y^δ with

$$\|y - y^\delta\| \leq \delta. \tag{2}$$

The Eq. (1) is ill-posed, i.e., the solution is not continuously depending on the data. So one has to consider regularization method for approximating the solution $\hat{x}$ of the

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Eq. (1). In the most commonly used Tikhonov regularization method, the minimizer x_α^δ of

$$J_\alpha(x) = \|Ax - y^\delta\|^2 + \alpha\,\|x\|^2, \tag{3}$$

is used as an approximation for $\hat{x}$. If the operator A in the Eq. (1) is positive self-adjoint, then Lavrentiev regularization or Simplified regularization or Ritz regularization method [3, 12, 23, 36, 37, 41, 43] is used. In this method, the minimizer w_α^δ of

$$J_\alpha(x) := \langle Ax, x\rangle - 2\,\langle y^\delta, x\rangle + \alpha\,\langle x, x\rangle, \quad \forall\ \alpha > 0, \tag{4}$$

is used as an approximation for $\hat{x}$.

Note that x_α^δ satisfies

$$x_\alpha^\delta = (A^*A + \alpha I)^{-1} A^* y^\delta,$$

and the minimizer w_α^δ of (4) satisfies

$$w_\alpha^\delta = (A + \alpha I)^{-1} y^\delta. \tag{5}$$

Observe that, if $A = A^*$ is a positive self-adjoint operator, then we have

$$x_\alpha^\delta = (A^2 + \alpha I)^{-1} A y^\delta. \tag{6}$$

One can see that (6) involves more computation than (5), so in general Lavrentiev regularization is used when $A = A^*$ is a positive self-adjoint operator.

The solution of (3) and the solution of (4) oversmoothen the solution $\hat{x}$ (see [17]). The Fractional Tikhonov regularization (FTR) method [9, 14, 17, 28] and the Fractional Lavrentiev regularization (FLR) method [15] were used to reduce the oversmoothing of the solution $\hat{x}$. In the FTR method, the minimizer $x_{\alpha,\beta}^\delta$ of

$$J_\alpha^\beta(x) = \|Ax - y^\delta\|_\beta^2 + \alpha\,\|x\|^2, \tag{7}$$

is considered to approximate $\hat{x}$. Here $\|x\|_\beta = \|(AA^*)^{(\beta-1)/4}x\|$ for some parameter $0 \le \beta \le 1$ (see [9, 14]). Reddy in [35] considered the Engl type discrepancy principles

$$G(\alpha, y^\delta) := \|\alpha\big((A^*A)^{\frac{\beta+1}{2}} + \alpha I\big)^{-1}\ (A^*A)^{\frac{\beta-1}{2}} A^* y^\delta\|^2 = \tau_1\,\frac{\delta^p}{\alpha^q}, \quad \forall\ \tau_1 > 0,$$

and

$$G_1(\alpha, y^\delta) := \|A^*Ax_{\alpha,\beta}^\delta - A^*y^\delta\|^2 = \frac{\delta^p}{\alpha^q}, \quad \forall\ p > 0\,, q > 0,\ \alpha > 0,$$

for choosing α for the fractional Tikhonov regularization method.

Later in [28], Morigi et al. further modified (7) by replacing the L^2-norm in (7) by $TV-$norm. Precisely, Morigi et al. [28] considered the minimizer of

$$J_\alpha^\beta(x) = \|Ax - y^\delta\|_\beta^2 + \alpha \|x\|_{TV}^2,$$

to approximate $\hat{x}$, when A is an $m \times n$ real matrix. In [17], Klann and Ramlau considered

$$x_{\alpha,\gamma}^\delta = (A^*A + \alpha I)^{-\gamma}(A^*A)^{\gamma-1}A^*y^\delta \;\; \text{forsome } \gamma > \frac{1}{2},$$

to approximate $\hat{x}$.

If A is a positive semi-definite operator, then the minimizer $w_{\alpha,\beta}$ of the functional

$$J_\alpha^\beta(x) := \langle Ax, x\rangle - 2\langle y, x\rangle + \alpha \langle A^\beta x, x\rangle, \quad \forall\, \alpha > 0, \tag{8}$$

where $0 \le \beta < 1$ (to be made precise later), is taken as an approximation for $\hat{x}$ (see [16]). Note that the minimizer of (8) satisfies

$$(A + \alpha A^\beta)w_{\alpha,\beta} = y, \tag{9}$$

and so, if (8) has a minimizer with $y = y^\delta$, then we have

$$y^\delta \in R(A^\beta). \tag{10}$$

Note that, if $y^\delta = A^\beta z$ for some z, then (9) with y^δ in place of y takes the form

$$(A^{1-\beta} + \alpha I)w_{\alpha,\beta} = z,$$

and $A^{1-\beta} + \alpha I$ is invertible. In other words, for $y^\delta \in R(A^\beta)$, (9) with y^δ has a unique solution. But the set of admissible data y^δ satisfying (10) is extremely thin (nowhere dense). This drawback can be overcome by considering finite dimensional realization of (9). Therefore, we consider the finite dimensional realization of (9).

Note that, in (3), (4) and (7), $\alpha > 0$ is called the regularization parameter and $\|x\|^2$ (or $\langle x, x\rangle$) is the penalty term. Observe that the minimizer $x_{\alpha,\beta}^\delta$ of (7) also satisfies

$$x_{\alpha,\beta}^\delta = \min_{x\in X}\{\|Ax - y^\delta\|^2 + \alpha\|(AA^*)^{\frac{1-\beta}{4}}x\|^2\},$$

and the penalty term $\|(AA^*)^{\frac{1-\beta}{4}}x\|^2$ minimizes the over smoothing. Similarly, the penalty term $\langle A^\beta x, x\rangle$ in (8) minimizes the over smoothing of the regularized solution. Natterer in 1984 [29] noticed the over smoothing of the Tikhonov regularization. The minimizer of (7) satisfies

$$((A^*A)^{1+\gamma} + \alpha I)x_{\alpha,\beta}^\delta = (A^*A)^\gamma A^*y^\delta, \tag{11}$$

where $\gamma = \frac{\beta-1}{2} \le 0$ (see [35]).

Even though, the FTR and FLR methods minimize the over smoothing occurred in the Standard Tikhonov regularization (STR) and Standard Lavrentiev regularization (SLR), the order of convergence for the FTR and FLR methods are smaller than that of STR and SLR. In order to improve the order of the convergence, one can study the fractional Tikhonov as well as the fractional simplified regularization methods in the setting of Hilbert scales [6–8, 11, 19, 21, 23, 24, 29–33, 39, 40] (see also Sect. 2).

Preliminaries are given in Sect. 2, the convergence analysis of the method is given in Sect. 3. Section 4, deals with comparison between Standard Lavrentiev and Fractional Lavrentiev regularization method in Hilbert scales, error bounds are given in Sect. 5, numerical experiments are given in Sect. 6 and the paper ends with a conclusion in Sect. 7.

2 Preliminaries

Let $A : \mathcal{X} \longrightarrow \mathcal{X}$ be an injective and positive self-adjoint operator defined on a real Hilbert space $\mathcal{X}$. We are concerned with the problem of approximating a solution $\hat{x}$ (assumed to exist) of the ill-posed equation

$$Ax = y. \tag{12}$$

As already mentioned in the introduction, our aim is to study the finite dimensional realization of the fractional Lavrentiev regularization method for approximately solving the Eq. (12) in the setting of Hilbert scales. So, let us first recall the definition of Hilbert scales:

Definition 1 [23] A family $\{\mathcal{X}_s\}_{s\in\mathbb{R}}$ of Hilbert spaces is called a Hilbert scale if it satisfies the following conditions:

(a) For $s < t$, $\mathcal{X}_t \subseteq \mathcal{X}_s$ and $\mathcal{X}_t$ is a dense subset of $\mathcal{X}_s$;
(b) As Hilbert spaces, the above inclusion is a continuous embedding, i.e., there exists $c_{s,t} > 0$ such that

$$\|x\|_s \leq c_{s,t}\|x\|_t, \quad \forall\, x \in \mathcal{X}_t. \tag{13}$$

Let $L : D(L) \subset \mathcal{X} \longrightarrow \mathcal{X}$, be a linear, unbounded, self-adjoint operator, which satisfies the following:

$$\langle Lx, x\rangle > 0, \quad \overline{D(L)} = \mathcal{X}, \quad \|Lx\| \geq \|x\|, \quad \forall\, x \in D(L).$$

Let $\mathcal{X}_t$ be the completion of $D := \cap_{k=0}^{\infty} D(L^k)$ with the norm $\|x\|_t = \|L^t x\|$, (here and below $\|\cdot\|$ denotes the norm in $\mathcal{X}$) induced by the inner product

$$\langle u, v\rangle_t = \langle L^t u, L^t v\rangle, \quad \forall\, u, v \in D.$$

Then $\{\mathcal{X}_s\}_{s\in\mathbb{R}}$ (see [6]) satisfies the Definition 1 [6, 7, 41, 43].

In this study, we consider the Hilbert scale $\{\mathcal{X}_s\}_{s\in\mathbb{R}}$. Note that the Hilbert scale generated by L connects $\mathcal{X}$ with $\mathcal{X}_s$ through the relation $\|x\|_s = \|x\|_{\mathcal{X}_s} = \|L^s x\|$ ([2], see also [19, Page 145]). We assume throughout the study that the operator A satisfies:

$$d_1\|x\|_{-a} \le \|Ax\| \le d_2\|x\|_{-a}, \quad x \in \mathcal{X} \tag{14}$$

for some $a > 0$, $d_1 > 0$ and $d_2 > 0$. Let $f(t) := \min\{d_1^t, d_2^t\}$, $g(t) := \max\{d_1^t, d_2^t\}$ for all $t \in \mathbb{R}$ and $|t| \le 1$.

Proposition 1 *c.f. ([29, Proposition 1]) Let $A : \mathcal{X} \longrightarrow \mathcal{X}$ be a bounded linear self-adjoint operator that satisfies (14). Then, for $|v| \le 1$,*

$$f(v)\|x\|_{-va} \le \|(A^*A)^{v/2}x\| \le g(v)\|x\|_{-va}, \quad \forall\, x \in D((A^*A)^{v/2}).$$

Remark 1 Note that the above proposition is valid for bounded linear operator A from a Hilbert space X into a Hilbert space Y.

For $|\tau| \le 2$, let $F(t) := \min\{f(\frac{\tau}{2})^t, g(\frac{\tau}{2})^t\}$, $G(t) := \max\{f(\frac{\tau}{2})^t, g(\frac{\tau}{2})^t\}$. Using the above Proposition 1, and notation, we prove the following proposition, which will be used extensively in our analysis.

Proposition 2 *Let $A : \mathcal{X} \longrightarrow \mathcal{X}$ be a bounded linear self-adjoint operator satisfying (14). Then we have the following:*

(1) For all $x \in D(A^v)$ and $|v| \le 1$,

$$f(v)\|x\|_{-va} \le \|A^v x\| \le g(v)\|x\|_{-va}.$$

(2) For all $x \in D(A^{\tau/2}L^{-s/2})$, $s > 0$ and $|\tau| \le 2$,

$$f(\frac{\tau}{2})\|x\|_{-\frac{\tau a+s}{2}} \le \|A^{\frac{\tau}{2}}L^{-s/2}x\| \le g(\frac{\tau}{2})\|x\|_{-\frac{\tau a+s}{2}}.$$

(3) For all $x \in D((L^{-s/2}A^\tau L^{-s/2})^{v/2})$, $s > 0$, $|\tau| \le 2$ and $|v| \le 1$,

$$F(v)\|x\|_{-v(\frac{\tau a+s}{2})} \le \|(L^{-s/2}A^\tau L^{-s/2})^{v/2}x\| \le G(v)\|x\|_{-v(\frac{\tau a+s}{2})}.$$

Proof Proof of (1) follows from Proposition 1 since $A^*A = A^2$. Note that, if we take $v = \tau/2$ in (1), then we obtain

$$f\left(\frac{\tau}{2}\right)\|x\|_{-\frac{\tau a}{2}} \le \|A^{\frac{\tau}{2}}x\| \le g\left(\frac{\tau}{2}\right)\|x\|_{-\frac{\tau a}{2}}, \quad \forall\, x \in D\left(A^{\frac{\tau}{2}}\right). \tag{15}$$

By taking $x = L^{-s/2}x$ in (15), we obtain (2). The proof of (3) follows by taking $A = A^{\tau/2}L^{-s/2}$ in Proposition 1. □

Let $\{P_h\}_{h>0}$ be a family of orthogonal projections of X onto $R(P_h)$, which is the range of P_h. Let

$$\epsilon_h := \|A(I - P_h)\| \longrightarrow 0 \text{ as } h \longrightarrow 0.$$

Let $A_h := P_h A P_h$. Then we have $\|(A_h - A)x\| \le \|P_h A(P_h - I)x\| + \|(P_h - I)Ax\|$. Therefore we assume that

$$\|(A_h - A)x\| \longrightarrow 0, \tag{16}$$

as $h \longrightarrow 0$. This condition is satisfied if, for example A is a compact operator and $P_h \longrightarrow I$ pointwise. So, let $h_0 > 0$ be such that

$$\|(A_h - A)x\| \le \tilde{\epsilon}_h = \frac{d_1\|x\|_{-a}}{2}, \quad \forall\, x \neq 0,\ h \le h_0. \tag{17}$$

Hereafter, we assume that $h \le h_0$. Let $\bar{d}_1 = \frac{d_1}{2}$ and $\bar{d}_2 = d_2 + \frac{d_1}{2}$. Using the above notation, we have the following lemma:

Lemma 3 *Let $\bar{d}_1$ and $\bar{d}_2$ be as above. Then, for all $P_h x \neq 0$ we have*

$$\bar{d}_1\|x\|_{-a} \le \|A_h x\| \le \bar{d}_2\|x\|_{-a}. \tag{18}$$

Proof Using (16) and (17), we have

$$\begin{aligned}\|A_h x\| &\le \|Ax\| + \|(A_h - A)x\| \\ &\le d_2\|x\|_{-a} + \tilde{\epsilon}_h \\ &\le \bar{d}_2\|x\|_{-a},\end{aligned}$$

and

$$\begin{aligned}\|A_h x\| &\ge \|Ax\| - \|(A_h - A)x\| \\ &\ge d_1\|x\|_{-a} - \tilde{\epsilon}_h \\ &\ge \bar{d}_1\|x\|_{-a}.\end{aligned}$$

□

Let $\bar{f}(t) := \min\{\bar{d}_1^t, \bar{d}_2^t\}$ and $\bar{g}(t) := \max\{\bar{d}_1^t, \bar{d}_2^t\}$ for all $t \in \mathbb{R}$ and $|t| \le 1$. Analogously to the proof of [29, Proposition 1], one can prove the following proposition:

Proposition 4 *Let $A_h : \mathcal{X} \longrightarrow \mathcal{X}$ be a bounded linear self-adjoint operator that satisfies (18). Then, for $P_h x \neq 0$ and $|v| \le 1$, we have*

$$\bar{f}(v)\|x\|_{-va} \le \|(A_h^* A_h)^{v/2} x\| \le \bar{g}(v)\|x\|_{-va}, \quad x \in D((A_h^* A_h)^{v/2}).$$

For $|\tau| \le 2$, let

$$\bar{F}(t) := \min\left\{\bar{f}(\frac{\tau}{2})^t, \bar{g}(\frac{\tau}{2})^t\right\}, \quad \bar{G}(t) := \max\left\{\bar{f}(\frac{\tau}{2})^t, \bar{g}(\frac{\tau}{2})^t\right\}.$$

Using the above Proposition 4, above notation and the proof of Proposition 2, one can prove the following proposition:

Proposition 5 *Let $A_h : \mathcal{X} \longrightarrow \mathcal{X}$ be a bounded linear self-adjoint operator satisfying (18). Then, for $P_h x \neq 0$, we have*

(1) For all $x \in D(A_h^{\nu})$ and $|\nu| \leq 1$,

$$\bar{f}(\nu)\|x\|_{-\nu a} \leq \|A_h^{\nu}x\| \leq \bar{g}(\nu)\|x\|_{-\nu a}.$$

(2) For all $x \in D(A_h^{\tau/2}L^{-s/2})$, $s > 0$ and $|\tau| \leq 2$,

$$\bar{f}\left(\frac{\tau}{2}\right)\|x\|_{-\frac{\tau a+s}{2}} \leq \|A_h^{\frac{\tau}{2}}L^{-s/2}x\| \leq \bar{g}\left(\frac{\tau}{2}\right)\|x\|_{-\frac{\tau a+s}{2}}.$$

(3) For all $x \in D((L^{-s/2}A_h^{\tau}L^{-s/2})^{\nu/2})$, $s > 0$, $|\tau| \leq 2$ and $|\nu| \leq 1$,

$$\bar{F}(\nu)\|x\|_{-\nu\left(\frac{\tau a+s}{2}\right)} \leq \|(L^{-s/2}A_h^{\tau}L^{-s/2})^{\nu/2}x\| \leq \bar{G}(\nu)\|x\|_{-\nu\left(\frac{\tau a+s}{2}\right)}.$$

3 Finite dimensional realization of FLR in Hilbert scales

Consider the minimizer $w_{\alpha,\beta}^{s}$ of the following functional:

$$J_{\alpha,\beta}^{s}(x) = \langle Ax, x\rangle - 2\langle y, x\rangle + \alpha\langle A^{\beta}x, x\rangle_{\frac{s}{2}}, \quad \forall\ \alpha > 0,\ s > 0, \tag{19}$$

where $0 \leq \beta \leq 1$ (to be precised later), as an approximation for $\hat{x}$. Note that the minimizer $w_{\alpha,\beta}^{s}$ satisfies the equation

$$(A + \alpha A^{\beta}L^{s})w_{\alpha,\beta}^{s} = y. \tag{20}$$

For $\beta = s = 0$, (20) is Lavrentiev regularization of (12). Throughout this study we assume that the available data y^{δ} satisfies (2), in this case instead of (20) we consider $w_{\alpha,\beta,h}^{s,\delta}$ satisfying the equation:

$$(A_h + \alpha A_h^{\beta}L^{s})w_{\alpha,\beta,h}^{s,\delta} = P_h y^{\delta}, \tag{21}$$

as an approximation for $\hat{x}$. Let

$$B_{\beta,s} := L^{-s/2}A^{1-\beta}L^{-s/2}$$

and

$$B_{\beta,s,h} := L^{-s/2}A_h^{1-\beta}L^{-s/2}.$$

Then we have

$$w_{\alpha,\beta}^{s} = L^{-s/2}(B_{\beta,s} + \alpha I)^{-1}L^{-s/2}A^{-\beta}y, \tag{22}$$

$$w_{\alpha,\beta,h}^{s} := L^{-s/2}(B_{\beta,s,h} + \alpha I)^{-1}L^{-s/2}A_h^{-\beta}P_h y \tag{23}$$

and

$$w_{\alpha,\beta,h}^{s,\delta} = L^{-s/2}(B_{\beta,s,h} + \alpha I)^{-1} L^{-s/2} A_h^{-\beta} P_h y^\delta. \tag{24}$$

Since $B_{\beta,s}$ and $B_{\beta,s,h}$ for all $s > 0$ are self-adjoint operators, we have

$$\|(B_{\beta,s} + \alpha I)^{-1} B_{\beta,s}^{\mu}\| \leq \alpha^{\mu-1}, \quad \forall\ 0 \leq \mu \leq 1,\ \alpha > 0, \tag{25}$$

and

$$\|(B_{\beta,s,h} + \alpha I)^{-1} B_{\beta,s,h}^{\mu}\| \leq \alpha^{\mu-1}, \quad \forall\ 0 \leq \mu \leq 1,\ \alpha > 0. \tag{26}$$

Lemma 6 *Let $w_{\alpha,\beta,h}^{s}$ and $w_{\alpha,\beta,h}^{s,\delta}$ be as in* (23) *and* (24), *respectively. Let A satisfy* (14) *and* (17) *holds. Then, for $0 \leq \beta \leq \frac{2s+a}{3a}$, $s \leq a$ we have*

$$\|w_{\alpha,\beta,h}^{s} - w_{\alpha,\beta,h}^{s,\delta}\| \leq \varphi(s,a,\beta,h)\alpha^{\frac{-a}{(1-\beta)a+s}}\delta,$$

where $\varphi(s,a,\beta,h) := \dfrac{\bar{G}\left(\frac{-(s-2\beta a)}{(1-\beta)a+s}\right)}{\bar{F}\left(\frac{s}{(1-\beta)a+s}\right)\bar{f}(\beta)}$.

Proof By (23) and (24), we have

$$\begin{aligned}\|w_{\alpha,\beta,h}^{s,\delta} - w_{\alpha,\beta,h}^{s}\| =& \|L^{-s/2}(B_{\beta,s,h} + \alpha I)^{-1} L^{-s/2} A_h^{-\beta} P_h(y^\delta - y)\| \\ =& \|(B_{\beta,s,h} + \alpha I)^{-1} L^{-s/2} A_h^{-\beta} P_h(y^\delta - y)\|_{-s/2}.\end{aligned}$$

So, from Proposition 5 (3) with

$$v = \frac{s}{(1-\beta)a+s}, \quad \tau = 1-\beta, \quad x = (B_{\beta,s,h} + \alpha I)^{-1} L^{-s/2} A_h^{-\beta} P_h(y^\delta - y)$$

and (26), we obtain in turn that

$$\begin{aligned}&\|w_{\alpha,\beta,h}^{s,\delta} - w_{\alpha,\beta,h}^{s}\| \\ &\leq \frac{1}{\bar{F}\left(\frac{s}{(1-\beta)a+s}\right)} \|B_{\beta,s,h}^{\frac{s}{2[(1-\beta)a+s]}}(B_{\beta,s,h} + \alpha I)^{-1} L^{-s/2} A_h^{-\beta} P_h(y^\delta - y)\| = \frac{1}{\bar{F}\left(\frac{s}{(1-\beta)a+s}\right)} \\ &\times \|B_{\beta,s,h}^{\frac{s-\beta a}{(1-\beta)a+s}}(B_{\beta,s,h} + \alpha I)^{-1} B_{\beta,s,h}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}} L^{-s/2} A_h^{-\beta} P_h(y^\delta - y)\| \\ &\leq \frac{1}{\bar{F}\left(\frac{s}{(1-\beta)a+s}\right)} \|B_{\beta,s,h}^{\frac{s-\beta a}{(1-\beta)a+s}}(B_{\beta,s,h} + \alpha I)^{-1}\| \\ &\times \|B_{\beta,s,h}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}} L^{-s/2} A_h^{-\beta} P_h(y^\delta - y)\| \\ &\leq \frac{1}{\bar{F}\left(\frac{s}{(1-\beta)a+s}\right)} \alpha^{\frac{-a}{(1-\beta)a+s}} \|B_{\beta,s,h}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}} L^{-s/2} A_h^{-\beta} P_h(y^\delta - y)\|.\end{aligned} \tag{27}$$

Next, we prove the following:

$$\left\| B_{\beta,s,h}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}} L^{-s/2} A_h^{-\beta} P_h(y^\delta - y) \right\| \leq \frac{\bar{G}\left(\frac{-(s-2\beta a)}{(1-\beta)a+s}\right)}{\bar{f}(\beta)} \delta. \tag{28}$$

Take

$$v = \frac{-(s-2\beta a)}{(1-\beta)a+s}, \quad \tau = 1-\beta, \quad x = L^{-s/2} A_h^{-\beta} P_h(y^\delta - y)$$

in Proposition 5 (3) and Proposition 5 (1) with $v = \beta$, we obtain

$$\begin{aligned}
&\| B_{\beta,s,h}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}} L^{-s/2} A_h^{-\beta} P_h(y^\delta - y) \| \\
&\quad \leq \bar{G}\left(\frac{-(s-2\beta a)}{(1-\beta)a+s}\right) \| L^{-s/2} A_h^{-\beta} P_h(y^\delta - y) \|_{s/2-\beta a} \\
&\quad = \bar{G}\left(\frac{-(s-2\beta a)}{(1-\beta)a+s}\right) \| A_h^{-\beta} P_h(y^\delta - y) \|_{-\beta a} \\
&\quad \leq \frac{\bar{G}\left(\frac{-(s-2\beta a)}{(1-\beta)a+s}\right)}{\bar{f}(\beta)} \| y^\delta - y \| \\
&\quad \leq \frac{\bar{G}\left(\frac{-(s-2\beta a)}{(1-\beta)a+s}\right)}{\bar{f}(\beta)} \delta.
\end{aligned}$$

□

Now, we will make use of the following formula ([18, Page 287]) to estimate $\| w_{\alpha,\beta,h}^s - w_{\alpha,\beta}^s \|$:

$$\begin{aligned}
B^z x = {} & \frac{\sin \pi z}{\pi} \int_0^\infty t^z \left[(B+tI)^{-1} x - \frac{\theta(t)}{t} x + \ldots + (-1)^n \frac{\theta(t)}{t^n} B^{n-1} x \right] dt \\
& + \frac{\sin \pi z}{\pi} \left[\frac{x}{z} - \frac{Bx}{z-1} + \ldots + (-1)^{n-1} \frac{B^{n-1} x}{z-n+1} \right], \quad \forall\, x \in \mathcal{X},
\end{aligned}$$

where

$$\theta(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq 1, \\ 1, & \text{if } 1 < t \leq \infty \end{cases}$$

for any positive self-adjoint operator B and for any complex number z such that $0 < Re\, z < n$. Taking $z = 1-\beta$ and $0 \leq \beta \leq 1$, we have by the above formula, for any $Z \in \mathcal{X}$,

$$[(A_h^2)^{\frac{1-\beta}{2}} - (A^2)^{\frac{1-\beta}{2}}]Z = \frac{\sin \pi(1-\beta)}{\pi} \int_0^\infty \lambda^{\frac{1-\beta}{2}} (A_h^2 + \lambda I)^{-1}(A^2 - A_h^2)(A^2 + \lambda I)^{-1} Z d\lambda. \tag{29}$$

The following assumption is used to estimate $\|\hat{x} - w^s_{\alpha,\beta}\|$.

Assumption 1 There exist some $E > 0$ and $0 < t \leq (1-\beta)\frac{a}{2} + s$ such that

$$\hat{x} \in M_{t,E} = \{x \in \mathcal{X} : \|x\|_t \leq E\}.$$

Lemma 7 *Let $w^s_{\alpha,\beta,h}, w^s_{\alpha,\beta}$ be as in (23) and (22), respectively. Let A satisfy (14) and (17) hold. Then,*

$$\|w^s_{\alpha,\beta,h} - w^s_{\alpha,\beta}\| \leq \varphi_1(s,a,\beta,h)\alpha^{\frac{-a}{(1-\beta)a+s}}\varepsilon_h,$$

where $\varphi_1(s,a,\beta,h) := \frac{1}{\bar{F}(\frac{2s}{(1-\beta)a+s})} \bar{G}\left(\frac{2\beta a}{(1-\beta)a+s}\right) \frac{1}{\bar{f}(\beta)} \|\hat{x}\| + C_h$, with

$$C_h = 8 \frac{\bar{G}(\frac{-s+2\beta a}{(1-\beta)a+s})}{\bar{F}(\frac{s}{(1-\beta)a+s})\bar{f}(\beta)} \frac{\sin \pi(\frac{1-\beta}{2})}{\pi} \left[\frac{a}{t} \frac{g(\frac{-t}{a})G(\frac{s-2t}{(1-\beta)a+s})}{F(\frac{s-2t}{(1-\beta)a+s})} \|\hat{x}\|_t + 2\|A\| \frac{G(\frac{s-2t}{(1-\beta)a+s})}{F(\frac{s-2t}{(1-\beta)a+s})} \|\hat{x}\|\right].$$

Proof Note that

$$\begin{aligned}
w^s_{\alpha,\beta,h} =& (A_h^{1-\beta} + \alpha L^s)^{-1} A_h^{-\beta} P_h y \\
=& (A_h^{1-\beta} + \alpha L^s)^{-1} A_h^{-\beta} P_h A \hat{x} \\
=& (A_h^{1-\beta} + \alpha L^s)^{-1} A_h^{1-\beta} \hat{x} + (A_h^{1-\beta} + \alpha L^s)^{-1} A_h^{-\beta} P_h A (I - P_h)\hat{x}, \\
w^s_{\alpha,\beta} =& (A^{1-\beta} + \alpha L^s)^{-1} A^{-\beta} y \\
=& (A^{1-\beta} + \alpha L^s)^{-1} A^{1-\beta} \hat{x}
\end{aligned}$$

and hence

$$\begin{aligned}
w^s_{\alpha,\beta,h} - w^s_{\alpha,\beta} =& [(A_h^{1-\beta} + \alpha L^s)^{-1} A_h^{1-\beta} - (A^{1-\beta} + \alpha L^s)^{-1} A^{1-\beta}]\hat{x} \\
&+ (A_h^{1-\beta} + \alpha L^s)^{-1} A_h^{-\beta} P_h A (I - P_h)\hat{x}.
\end{aligned}$$

So

$$\|w^s_{\alpha,\beta,h} - w^s_{\alpha,\beta}\| \leq \|\grave{e}\| + \|(A_h^{1-\beta} + \alpha L^s)^{-1} A_h^{-\beta} P_h A (I - P_h)\hat{x}\|, \tag{30}$$

where $\grave{e} = [(A_h^{1-\beta} + \alpha L^s)^{-1} A_h^{1-\beta} - (A^{1-\beta} + \alpha L^s)^{-1} A^{1-\beta}]\hat{x}$.

Further, we have

$$\begin{aligned}
&\|(A_h^{1-\beta}+\alpha L^s)^{-1}A_h^{-\beta}P_hA(I-P_h)\hat{x}\| \\
&\quad\leq \|L^{-\frac{s}{2}}(B_{\beta,s,h}+\alpha I)^{-1}L^{-\frac{s}{2}}A_h^{-\beta}P_hA(I-P_h)\hat{x}\| \\
&\quad\leq \frac{1}{\bar{F}(\frac{s}{(1-\beta)a+s})}\|(B_{\beta,s,h}+\alpha I)^{-1}B_{\beta,s,h}^{\frac{s-\beta a}{(1-\beta)a+s}}B_{\beta,s,h}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}}L^{\frac{-s}{2}}A_h^{-\beta}P_hA(I-P_h)\hat{x}\| \\
&\leq \frac{1}{\bar{F}(\frac{s}{(1-\beta)a+s})}\alpha^{\frac{-a}{(1-\beta)a+s}} \\
&\quad\times\|B_{\beta,s,h}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}}L^{\frac{-s}{2}}A_h^{-\beta}P_hA(I-P_h)\hat{x}\| \\
&\quad\leq \frac{1}{\bar{F}(\frac{s}{(1-\beta)a+s})}\alpha^{\frac{-a}{(1-\beta)a+s}}\bar{G}\left(\frac{2\beta a-s}{(1-\beta)a+s}\right)\|A_h^{-\beta}P_hA(I-P_h)\hat{x}\|_{-\beta a} \\
&\quad\leq \frac{1}{\bar{F}(\frac{s}{(1-\beta)a+s})}\alpha^{\frac{-a}{(1-\beta)a+s}}\bar{G}\left(\frac{2\beta a-s}{(1-\beta)a+s}\right)\frac{1}{\bar{f}(\beta)}\|P_hA(I-P_h)\hat{x}\| \\
&\quad\leq \frac{1}{\bar{F}(\frac{s}{(1-\beta)a+s})}\bar{G}\left(\frac{2\beta a-s}{(1-\beta)a+s}\right)\frac{1}{\bar{f}(\beta)}\epsilon_h\|\hat{x}\|\alpha^{\frac{-a}{(1-\beta)a+s}},
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
\grave{e} =&[L^{-s/2}(B_{\beta,s,h}+\alpha I)^{-1}B_{\beta,s,h}-L^{-s/2}(B_{\beta,s}+\alpha I)^{-1}B_{\beta,s}]L^{\frac{s}{2}}\hat{x} \\
=&L^{-s/2}(B_{\beta,s,h}+\alpha I)^{-1}[B_{\beta,s,h}(B_{\beta,s}+\alpha I)-(B_{\beta,s,h}+\alpha I)B_{\beta,s}](B_{\beta,s}+\alpha I)^{-1}L^{\frac{s}{2}}\hat{x} \\
=&L^{-s/2}(B_{\beta,s,h}+\alpha I)^{-1}\alpha[B_{\beta,s,h}-B_{\beta,s}](B_{\beta,s}+\alpha I)^{-1}L^{\frac{s}{2}}\hat{x} \\
=&L^{-s/2}(B_{\beta,s,h}+\alpha I)^{-1}\alpha[L^{-s/2}A_h^{1-\beta}L^{-s/2}-L^{-s/2}A^{1-\beta}L^{-s/2}](B_{\beta,s}+\alpha I)^{-1}L^{\frac{s}{2}}\hat{x} \\
=&L^{-s/2}(B_{\beta,s,h}+\alpha I)^{-1}L^{-s/2}(A_h^{1-\beta}-A^{1-\beta})\alpha L^{-\frac{s}{2}}(B_{\beta,s}+\alpha I)^{-1}L^{\frac{s}{2}}\hat{x} \\
=&L^{-s/2}(B_{\beta,s,h}+\alpha I)^{-1}L^{-s/2}((A_h^2)^{\frac{1-\beta}{2}}-(A^2)^{\frac{1-\beta}{2}})\alpha L^{-\frac{s}{2}}(B_{\beta,s}+\alpha I)^{-1}L^{\frac{s}{2}}\hat{x}
\end{aligned}$$

so by (29), we have

$$\begin{aligned}
\grave{e} =&\frac{\sin\pi(\frac{1-\beta}{2})}{\pi}L^{-s/2}(B_{\beta,s,h}+\alpha I)^{-1} \\
&\times\int_0^\infty \lambda^{\frac{1-\beta}{2}}L^{-s/2}(A_h^2+\lambda I)^{-1}(A^2-A_h^2)(A^2+\lambda I)^{-1}\alpha Z d\lambda
\end{aligned}$$

where $Z=L^{-s/2}(B_{\beta,s}+\alpha I)^{-1}L^{s/2}\hat{x}$. Therefore,

$$
\begin{aligned}
\|è\| &\leq \frac{1}{\bar{F}(\frac{s}{(1-\beta)a+s})}\frac{\sin\pi(\frac{1-\beta}{2})}{\pi}\|B_{\beta,s,h}^{\frac{s}{2[(1-\beta)a+s]}}(B_{\beta,s,h}+\alpha I)^{-1}\\
&\quad\times\int_0^\infty \lambda^{\frac{1-\beta}{2}}L^{-s/2}(A_h^2+\lambda I)^{-1}(A^2-A_h^2)(A^2+\lambda I)^{-1}\alpha Z\|d\lambda\\
&\leq \frac{1}{\bar{F}(\frac{s}{(1-\beta)a+s})}\frac{\sin\pi(\frac{1-\beta}{2})}{\pi}\alpha\|B_{\beta,s,h}^{\frac{2s-2\beta a}{2[(1-\beta)a+s]}}(B_{\beta,s,h}+\alpha I)^{-1}\\
&\quad\times\int_0^\infty \lambda^{\frac{1-\beta}{2}}B_{\beta,s,h}^{\frac{-s+2\beta a}{2[(1-\beta)a+s]}}L^{-s/2}(A_h^2+\lambda I)^{-1}(A^2-A_h^2)(A^2+\lambda I)^{-1}Z\|d\lambda\\
&\leq \frac{1}{\bar{F}(\frac{s}{(1-\beta)a+s})}\frac{\sin\pi(\frac{1-\beta}{2})}{\pi}\alpha^{\frac{2s-2\beta a}{2[(1-\beta)a+s]}}\\
&\quad\times\int_0^\infty \lambda^{\frac{1-\beta}{2}}\|B_{\beta,s,h}^{\frac{-s+2\beta a}{2[(1-\beta)a+s]}}L^{-s/2}(A_h^2+\lambda I)^{-1}(A^2-A_h^2)(A^2+\lambda I)^{-1}Z\|d\lambda\\
&\leq \frac{\bar{G}(\frac{-s+2\beta a}{(1-\beta)a+s})}{\bar{F}(\frac{s}{(1-\beta)a+s})}\frac{\sin\pi(\frac{1-\beta}{2})}{\pi}\alpha^{\frac{s-\beta a}{(1-\beta)a+s}}\\
&\quad\times\int_0^\infty \lambda^{\frac{1-\beta}{2}}\|L^{-s/2}(A_h^2+\lambda I)^{-1}(A^2-A_h^2)(A^2+\lambda I)^{-1}Z\|_{\frac{s}{2}-\beta a}\ d\lambda\\
&\leq \frac{\bar{G}(\frac{-s+2\beta a}{(1-\beta)a+s})}{\bar{F}(\frac{s}{(1-\beta)a+s})\bar{f}(\beta)}\frac{\sin\pi(\frac{1-\beta}{2})}{\pi}\alpha^{\frac{s-\beta a}{(1-\beta)a+s}}\\
&\quad\times\int_0^\infty \lambda^{\frac{1-\beta}{2}}\|(A_h^2)^{\frac{\beta}{2}}(A_h^2+\lambda I)^{-1}[A_h(A-A_h)+(A-A_h)A](A^2+\lambda I)^{-1}Z\|d\lambda\\
&\leq \frac{\bar{G}(\frac{-s+2\beta a}{(1-\beta)a+s})}{\bar{F}(\frac{s}{(1-\beta)a+s})\bar{f}(\beta)}\frac{\sin\pi(\frac{1-\beta}{2})}{\pi}[L_1+L_2]
\end{aligned}
\tag{32}
$$

where $L_1=\alpha^{\frac{s-\beta a}{(1-\beta)a+s}}\int_0^\infty \lambda^{\frac{1-\beta}{2}}\|(A_h^2)^{\frac{\beta}{2}}(A_h^2+\lambda I)^{-1}[A_h(A-A_h)](A^2+\lambda I)^{-1}Z\|d\lambda$ and $L_2=\alpha^{\frac{s-\beta a}{(1-\beta)a+s}}\int_0^\infty \lambda^{\frac{1-\beta}{2}}\|(A_h^2)^{\frac{\beta}{2}}(A_h^2+\lambda I)^{-1}[(A-A_h)A](A^2+\lambda I)^{-1}Z\|d\lambda$.

$$\begin{aligned}
L_1 &\le \alpha^{\frac{s-\beta a}{(1-\beta)a+s}} \int_0^1 \lambda^{\frac{1-\beta}{2}} \|(A_h^2)^{\frac{\beta}{2}}(A_h^2+\lambda I)^{-1}A_h\| \|(A-A_h)\| \|(A^2+\lambda I)^{-1}A^{\frac{t}{a}}\| \|A^{-\frac{t}{a}}Z\| d\lambda \\
&\quad + \alpha^{\frac{s-\beta a}{(1-\beta)a+s}} \int_1^\infty \lambda^{\frac{1-\beta}{2}} \|(A_h^2)^{\frac{\beta}{2}}(A_h^2+\lambda I)^{-1}\| \|A_h\| \|(A-A_h)\| \|(A^2+\lambda I)^{-1}Z\| d\lambda \\
&\le \alpha^{\frac{s-\beta a}{(1-\beta)a+s}} \int_0^1 \lambda^{\frac{t}{2a}-1} 2\varepsilon_h \|A^{-\frac{t}{a}}Z\| d\lambda \\
&\quad + \alpha^{\frac{s-\beta a}{(1-\beta)a+s}} \int_1^\infty \frac{\|A_h\| 2\varepsilon_h \|Z\|}{\lambda^{\frac{3}{2}}} d\lambda \\
&\le \alpha^{\frac{s-\beta a}{(1-\beta)a+s}} \left[\frac{2a}{t} 2\varepsilon_h \|A^{-\frac{t}{a}}Z\| + 4\|A_h\| 2\varepsilon_h \|Z\| \right].
\end{aligned} \tag{33}$$

Further, observe that

$$\begin{aligned}
\|A^{\frac{-t}{a}}Z\| &= \|A^{\frac{-t}{a}} L^{-s/2}(B_{\beta,s}+\alpha I)^{-1} L^{s/2}\hat{x}\| \\
&\le g(\frac{-t}{a}) \|(B_{\beta,s}+\alpha I)^{-1} L^{s/2}\hat{x}\|_{t-\frac{s}{2}} \\
&\le \frac{g(\frac{-t}{a})}{F(\frac{s-2t}{(1-\beta)a+s})} \|(B_{\beta,s}+\alpha I)^{-1} B_{\beta,s}^{\frac{s-2t}{2[(1-\beta)a+s]}} L^{s/2}\hat{x}\| \\
&\le \frac{g(\frac{-t}{a}) G(\frac{s-2t}{(1-\beta)a+s})}{F(\frac{s-2t}{(1-\beta)a+s})} \alpha^{-1} \|\hat{x}\|_t
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
\|Z\| &= \|L^{-s/2}(B_{\beta,s}+\alpha I)^{-1} L^{s/2}\hat{x}\| \\
&\le \frac{G(\frac{s-2t}{(1-\beta)a+s})}{F(\frac{s-2t}{(1-\beta)a+s})} \alpha^{-1} \|\hat{x}\|.
\end{aligned} \tag{35}$$

Therefore, from (33), (34) and (35) it follows that

$$L_1 \le 4\left[\frac{a}{t} \frac{g(\frac{-t}{a}) G(\frac{s-2t}{(1-\beta)a+s})}{F(\frac{s-2t}{(1-\beta)a+s})} \|\hat{x}\|_t + 2\|A_h\| \frac{G(\frac{s-2t}{(1-\beta)a+s})}{F(\frac{s-2t}{(1-\beta)a+s})} \|\hat{x}\| \right] \varepsilon_h \ \alpha^{\frac{-a}{(1-\beta)a+s}}. \tag{36}$$

Proceeding in a similar manner for L_2 we get

$$L_2 \le 4\left[\frac{a}{t} \frac{g(\frac{-t}{a}) G(\frac{s-2t}{(1-\beta)a+s})}{F(\frac{s-2t}{(1-\beta)a+s})} \|\hat{x}\|_t + 2\|A\| \frac{G(\frac{s-2t}{(1-\beta)a+s})}{F(\frac{s-2t}{(1-\beta)a+s})} \|\hat{x}\| \right] \varepsilon_h \ \alpha^{\frac{-a}{(1-\beta)a+s}}. \tag{37}$$

The result, follows from (30), (32) (36), (37) and the fact that $\|A_h\| \le \|A\|$. □

Lemma 8 *Let $w^s_{\alpha,\beta}$ be as in (22), A satisfies (14) and suppose that Assumption 1 holds. Then, for $0 < \beta \le \frac{2s+a}{3a}$, $s \le a$, we have*

$$\|\hat{x} - w^s_{\alpha,\beta}\| \leq \psi(s,a,\beta,t)\alpha^{\frac{t}{(1-\beta)a+s}},$$

where

$$\psi(s,a,\beta,t) := \frac{G\left(\frac{s-2t}{(1-\beta)a+s}\right)}{F\left(\frac{s}{(1-\beta)a+s}\right)}E.$$

Proof By Assumption 1 and (22), we have

$$\begin{aligned}\hat{x} - w^s_{\alpha,\beta} =& \hat{x} - (A^{1-\beta} + \alpha L^s)^{-1}A^{-\beta}y\\ =& \alpha(A^{1-\beta} + \alpha L^s)^{-1}L^s\hat{x}\\ =& \alpha L^{-s/2}(B_{\beta,s} + \alpha I)^{-1}L^{s/2}\hat{x},\end{aligned}$$

that is,

$$\|\hat{x} - w^s_{\alpha,\beta}\| = \alpha\|(B_{\beta,s} + \alpha I)^{-1}L^{s/2}\hat{x}\|_{-s/2}.$$

Thus, by Proposition 2 (3) (by taking $v = \frac{s}{(1-\beta)a+s}$) and (25), we have

$$\begin{aligned}\|\hat{x} - w^s_{\alpha,\beta}\| \leq& \frac{1}{F\left(\frac{s}{(1-\beta)a+s}\right)}\left\|\alpha B_{\beta,s}^{\frac{s}{2[(1-\beta)a+s]}}(B_{\beta,s} + \alpha I)^{-1}L^{s/2}\hat{x}\right\|\\ \leq& \frac{1}{F\left(\frac{s}{(1-\beta)a+s}\right)}\left\|\alpha B_{\beta,s}^{\frac{t}{(1-\beta)a+s}}(B_{\beta,s} + \alpha I)^{-1}\right\|\left\|B_{\beta,s}^{\frac{s-2t}{2[(1-\beta)a+s]}}L^{s/2}\hat{x}\right\|\\ \leq& \frac{G\left(\frac{s-2t}{(1-\beta)a+s}\right)}{F\left(\frac{s}{(1-\beta)a+s}\right)}\alpha^{\frac{t}{(1-\beta)a+s}}\|\hat{x}\|_t\\ \leq& \frac{G\left(\frac{s-2t}{(1-\beta)a+s}\right)}{F\left(\frac{s}{(1-\beta)a+s}\right)}\alpha^{\frac{t}{(1-\beta)a+s}}E.\end{aligned}$$

□

Combining Lemmas 6, 7 and 8, we obtain the following theorem:

Theorem 9 *Let $w^{s,\delta}_{\alpha,\beta,h}$ be as in (24), A satisfies (14) and suppose that Assumption 1 and (17) hold. Then, for $0 \leq \beta \leq \frac{2s+a}{3a}$, $s \leq a$ we have*

$$\|\hat{x} - w^{s,\delta}_{\alpha,\beta,h}\| \leq \varphi_2(s,a,\beta,h)\alpha^{\frac{-a}{(1-\beta)a+s}}(\delta + \epsilon_h) + \psi(s,a,\beta,t)\alpha^{\frac{t}{(1-\beta)a+s}},$$

where $\varphi_2(s,a,\beta,h) = \max\{\varphi_1(s,a,\beta,h), \varphi(s,a,\beta,h)\}$. In particular, if $\alpha := \alpha(s,a,\beta,h,t) = c_0(\delta + \epsilon_h)^{\frac{(1-\beta)a+s}{t+a}}$ for some $c_0 > 0$, then

$$\|\hat{x} - w^{s,\delta}_{\alpha,\beta,h}\| \leq \eta(s,a,\beta,t)(\delta + \epsilon_h)^{\frac{t}{t+a}},$$

where $\eta(s,a,\beta,h,t) = \max\left\{\varphi_2(s,a,\beta,h)c_0^{\frac{-a}{(1-\beta)a+s}}, \psi(s,a,\beta,t)c_0^{\frac{t}{(1-\beta)a+s}}\right\}$.

3.1 Order optimality

As in [26], we define the best possible worst error for identifying the solution $\hat{x}$ of (12) from $y^\delta \in \mathcal{X}$ satisfying (2) and $\hat{x}$ satisfying Assumption 1 as

$$\Theta(M_{t,E},\delta) = \inf_R \sup\{\|\hat{x} - Ry^\delta\| : \hat{x} \in M_{t,E},\, y^\delta \in \mathcal{X},\, \|A\hat{x} - y^\delta\| \leq \delta\}.$$

Here, the minimum is taken over all regularization methods $R : \mathcal{X} \longrightarrow \mathcal{X}$. Let

$$e(M_{t,E},\delta) := \sup\{\|x\| : x \in M_{t,E},\, \|Ax\| \leq \delta\}.$$

Then, since $\mathcal{X}$ is a Hilbert space and A is positive self-adjoint, we have (see [25]) $e(M_{t,E},\delta) = \Theta(M_{t,E},\delta)$.

A regularization method R_α with a parameter choice strategy $\alpha = \alpha(\delta)$ is said to be of optimal order if

$$\|R_\alpha y^\delta - \hat{x}\| = O(e(M_{t,E},\delta)).$$

Using the interpolation inequality (see [19]):

$$\|x\|_s \leq \|x\|_r^\theta \|x\|_t^{1-\theta}, \quad \forall x \in \mathcal{X}_t,$$

where $r \leq s \leq t$ and $\theta = \frac{t-s}{t-r}$ with $r = -a$ and $s = 0$, we obtain

$$\begin{aligned}
\|x\| &\leq \|x\|_{-a}^{\frac{t}{t+a}} \|x\|_t^{\frac{a}{t+a}} \\
&\leq \left(\frac{\|Ax\|}{d_1}\right)^{\frac{t}{t+a}} \|x\|_t^{\frac{a}{t+a}} \\
&\leq \left(\frac{\delta}{d_1}\right)^{\frac{t}{t+a}} \|x\|_t^{\frac{a}{t+a}}, \quad \forall, x \in M_{t,E},
\end{aligned}$$

and the above estimate is sharp (see [42]).

So, a regularization method is called optimal order yielding regularization method with respect to $M_{t,E}$ and (14) if it yields an approximation, say $R_\alpha y^\delta$ with $\|y - y^\delta\| \leq \delta$ and satisfies

$$\|R_\alpha y^\delta - \hat{x}\| = O(\delta^{\frac{t}{t+a}}).$$

Theorem 9 shows that we obtained the optimal order for the choice of $\alpha := \alpha(s,a,\beta,t) = c_0 \delta^{\frac{(1-\beta)a+s}{t+a}}$ for some $c_0 > 0$.

4 Standard Lavrentiev method vs FLR method in Hilbert scales

The filter factors [15] of the SLR method and the FLR method in the Hilbert scales are compared in this section. Recall [6–8, 22] that the Lavrentiev regularized solution for (12) in Hilbert scales is given by

$$w_\alpha^s = L^{-s/2}(L^{-s/2}AL^{-s/2}+\alpha I)^{-1}L^{-s/2}y. \tag{38}$$

So, using Proposition 2 (3) with $\tau = 1$, we have

$$\begin{aligned}\|w_\alpha^s\| &\leq \frac{1}{F(\frac{s}{s+a})}\|(L^{-s/2}AL^{-s/2})^{\frac{s}{2(s+a)}}(L^{-s/2}AL^{-s/2}+\alpha I)^{-1}L^{-s/2}y\| \\ &\leq \frac{1}{F(\frac{s}{s+a})}\|(L^{-s/2}AL^{-s/2})^{\frac{s}{(s+a)}}(L^{-s/2}AL^{-s/2}+\alpha I)^{-1} \\ &\quad \times\ (L^{-s/2}AL^{-s/2})^{\frac{-s}{2(s+a)}}L^{-s/2}y\|\end{aligned} \tag{39}$$

and hence

$$\begin{aligned}\|w_\alpha^s\|^2 \leq \frac{1}{F(\frac{s}{s+a})^2}&\int_0^{\|L^{-s/2}AL^{-s/2}\|}\left(\frac{\lambda^{\frac{s}{(s+a)}}}{\lambda+\alpha}\right)^2 \\ &\times\ d\langle E_\lambda(L^{-s/2}AL^{-s/2})^{\frac{-s}{2(s+a)}}L^{-s/2}y,(L^{-s/2}AL^{-s/2})^{\frac{-s}{2(s+a)}}L^{-s/2}y\rangle,\end{aligned} \tag{40}$$

where $\{E_\lambda : 0 \leq \lambda \leq \|L^{-s/2}AL^{-s/2}\|\}$ is the spectral family of $L^{-s/2}AL^{-s/2}$. Similarly, we have

$$\begin{aligned}\|w_{\alpha,\beta}^s\| \leq &\frac{1}{F\left(\frac{s}{(1-\beta)a+s}\right)} \\ &\times\ \|B_{\beta,s}^{\frac{2s-2\beta a}{2[(1-\beta)a+s]}}(B_{\beta,s}+\alpha I)^{-1}B_{\beta,s}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}}L^{-s/2}A^{-\beta}y\|\end{aligned} \tag{41}$$

and hence

$$\begin{aligned}\|w_{\alpha,\beta}^s\|^2 \leq &\frac{1}{F\left(\frac{s}{(1-\beta)a+s}\right)^2}\int_0^{\|B_{\beta,s}\|}\left(\frac{\lambda^{\frac{s-\beta a}{(1-\beta)a+s}}}{\lambda+\alpha}\right)^2 \\ &\times\ d\langle F_\lambda B_{\beta,s}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}}L^{-s/2}A^{-\beta}y, B_{\beta,s}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}}L^{-s/2}A^{-\beta}y\rangle,\end{aligned} \tag{42}$$

where $\{F_\lambda : 0 \leq \lambda \leq \|B_{\beta,s}\|\}$ is the spectral family of $B_{\beta,s}$. Further, note that

$$\begin{aligned} & d\langle E_\lambda (L^{-s/2}AL^{-s/2})^{\frac{-s}{2(s+a)}} L^{-s/2}y, (L^{-s/2}AL^{-s/2})^{\frac{-s}{2(s+a)}} L^{-s/2}y\rangle \\ & \quad = \|E_\lambda (L^{-s/2}AL^{-s/2})^{\frac{-s}{2(s+a)}} L^{-s/2}y\|^2 \\ & \quad \le \|(L^{-s/2}AL^{-s/2})^{\frac{-s}{2(s+a)}} L^{-s/2}y\|^2 \\ & \quad \le G^2\left(\frac{-s}{s+a}\right)\|y\|^2. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} & d\left\langle F_\lambda B_{\beta,s}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}} L^{-s/2}A^{-\beta}y, B_{\beta,s}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}} L^{-s/2}A^{-\beta}y\right\rangle \\ & \quad \le \left(\frac{G\left(\frac{-(s-2\beta a)}{(1-\beta)a+s}\right)}{f(\beta)}\right)^2 \|y\|^2. \end{aligned}$$

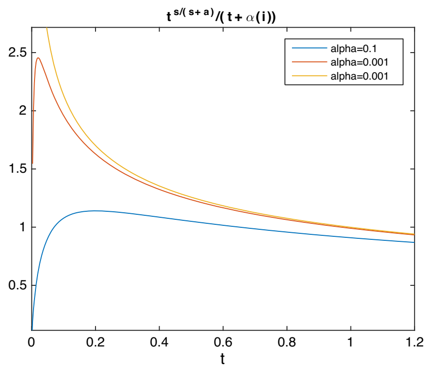
So, the quality of the the solution w_α^s and $w_{\alpha,\beta}^s$ are depending on the integrands in (40) and (42), respectively.

Let $\varphi_1(t) := \frac{t^{\frac{s}{(s+a)}}}{t+\alpha}$ and $\varphi_2(t) := \frac{t^{\frac{s-\beta a}{(1-\beta)a+s}}}{t+\alpha}$. The functions φ_1 and φ_2 are called the filter factors [15, 17] of the SLR method in Hilbert scales and the FLR method in Hilbert scales, respectively. Figure 1a shows the filter function $t \longrightarrow \varphi_1(t)$ for the SLR method in Hilbert scales. Figure 1b shows the filter function $t \longrightarrow \varphi_2(t)$ for the FLR method in Hilbert scales for $\beta = 0.5, 0.35, 0.25, 0.2, 0.1$.

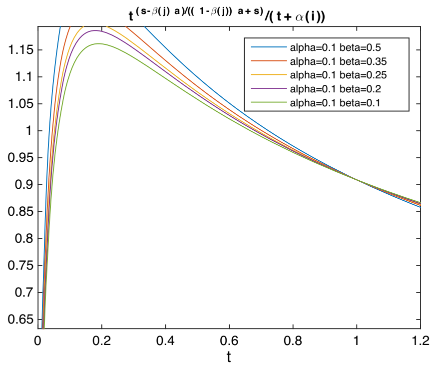
Note that one would like the filter functions to satisfy

$$\lim_{t\longrightarrow 0} \varphi_1(t) = 0 \text{ and } \lim_{t\longrightarrow 0} \varphi_2(t) = 0.$$

Note that (see Fig. 1a and b) the filter function $\varphi_2(t)$ is smoother than the filter function $\varphi_1(t)$ near 0.



(a) $\varphi_1(t)$ as a function of t.



(b) Filter function $\varphi_2(t)$ for $\alpha = 0.1$ and $\beta = 0.5, 0.35, 0.25, 0.2, 0.1$.

Fig. 1 Filter function $\varphi_2(t)$ plotted for different parameters

Remark 2 (cf. [1, Proposition 10]) Note that $\frac{\delta+\varepsilon_h}{\alpha^{\frac{a}{(1-\beta)a+s}}}$ is increasing for $\beta \in [0, \frac{s}{a}]$, whereas $\alpha^{\frac{t}{(1-\beta)+s}}$ (see Theorem 9) is decreasing for $\beta \in [0, \frac{s}{a}]$. Therefore, one has to choose $\beta \in [0, \frac{s}{a}]$ such that $\frac{\delta+\varepsilon_h}{\alpha^{\frac{a}{(1-\beta)a+s}}} = \alpha^{\frac{t}{(1-\beta)+s}}$ in order to obtain an optimal order error estimate for $\|\hat{x} - w^{s,\delta}_{\alpha,\beta,h}\|$. For a fixed $\delta > 0$, $t > 0$, $s > 0$, $a > 0$ and $\alpha \in [\delta^{\frac{s+a}{a+t}}, \delta^{\frac{a}{a+t}}]$, the best possible choice for β is

$$\beta = 1 + \frac{s}{a} - \left(\frac{a+t}{a}\right)\frac{\log\alpha}{\log(\delta+\varepsilon_h)}.$$

In this case, $\beta \in [0, \frac{s}{a}]$ and $\alpha = (\delta+\varepsilon_h)^{\frac{(1-\beta)a+s}{a+t}}$. But such a β and α is almost impossible in practice because t is unknown. So, in Sect. 5, we study the modified form of parameter choice strategy in [5].

5 Error bounds under a parameter choice strategy

In this section, we study the analogous of the discrepancy principle considered by George and Nair [5]. For a fixed $s \geq 0$, $\beta \geq 0$ and $\rho > -1$, let

$$F_{s,\beta,h}(\alpha, x) = \alpha^{\rho+1}\|(B_{\beta,s,h} + \alpha I)^{-(\rho+1)}L^{-s/2}x\|, \quad \forall\, x \in \mathcal{X},\ \alpha > 0.$$

Let $c > c_{\frac{-s}{2},0}$ and $y^\delta \in \mathcal{X}$ be such that

$$0 < c(\delta + \epsilon_h) \leq \|L^{-s/2}y^\delta\|. \tag{43}$$

Next, we prove the existence of a unique solution for

$$F_{s,\beta,h}(\alpha, y^\delta) = c(\delta + \epsilon_h).$$

Proposition 10 *Let c be as in (43). Then there exists a unique* $\alpha := \alpha(\delta, \beta, h, s, y^\delta) > 0$ *such that*

$$F_{s,\beta,h}(\alpha, y^\delta) = c(\delta + \epsilon_h). \tag{44}$$

Furthermore, we have

$$(c - c_{\frac{-s}{2},0})(\delta + \epsilon_h) \leq F_{s,\beta,h}(\alpha, y) \leq (c + c_{\frac{-s}{2},0})(\delta + \epsilon_h). \tag{45}$$

Proof Note that

$$
\begin{aligned}
&F^2_{s,\beta,h}(\alpha, y^\delta)\\
&\quad = \alpha^{2(\rho+1)}\langle (B_{\beta,s,h}+\alpha I)^{-(\rho+1)} L^{-s/2} y^\delta, (B_{\beta,s,h}+\alpha I)^{-(\rho+1)} L^{-s/2} y^\delta\rangle\\
&\quad = \int_0^{\|B_{\beta,s,h}\|} \left(\frac{\alpha}{\lambda+\alpha}\right)^{2(\rho+1)} d\langle F_\lambda L^{-s/2} y^\delta, L^{-s/2} y^\delta\rangle,
\end{aligned} \tag{46}
$$

where $\{F_\lambda : 0 \le \lambda \le \|B_{\beta,s,h}\|\}$ is the spectral family of $B_{\beta,s,h}$. Now, since the map $\alpha \longrightarrow \varphi(\alpha, \lambda) := \left(\frac{\alpha}{\lambda+\alpha}\right)^{2(\rho+1)}$ is strictly increasing for $\lambda > 0$, we have

$$
\varphi(\alpha, \lambda) \longrightarrow 0 \text{ as } \alpha \longrightarrow 0
$$

and

$$
\varphi(\alpha, \lambda) \longrightarrow 1 \text{ as } \alpha \longrightarrow \infty,
$$

by the Dominated Convergence Theorem, there exists a unique $\alpha := \alpha(\delta, \beta, h, s, y^\delta) > 0$ satisfying (44).

The second part of the proposition follows by noting that

$$
\begin{aligned}
F_{s,\beta,h}(\alpha, y) &\le F_{s,\beta,h}(\alpha, y - y^\delta) + F_{s,\beta,h}(\alpha, y^\delta)\\
&\le \|L^{-s/2}(y - y^\delta)\| + F_{s,\beta,h}(\alpha, y^\delta)\\
&\le (c_{\frac{-s}{2},0} + c)(\delta + \epsilon_h),
\end{aligned}
$$

and

$$
\begin{aligned}
F_{s,\beta,h}(\alpha, y) &\ge F_{s,\beta,h}(\alpha, y^\delta) - F_{s,\beta,h}(\alpha, y - y^\delta)\\
&\ge c(\delta + \epsilon_h) - \|L^{-s/2}(y - y^\delta)\|\\
&\ge (c - c_{\frac{-s}{2},0})(\delta + \epsilon_h),
\end{aligned}
$$

where we used the relation $\|L^{-s/2}x\| \le c_{\frac{-s}{2},0}\|x\|$ for all $x \in \mathcal{X}$. □

The proof of the following proposition is analogous to the proof of Proposition 3.5 in [5] and so details are ignored.

Proposition 11 *(see [5, Proposition 3.5]) Let y^δ satisfy (43) and $0 \ne y \in \mathcal{X}$. Let $(\delta + \epsilon_h) > 0$ and $\alpha := \alpha(\delta, \beta, h, s, y^\delta) > 0$ be chosen according to (44). Then there exists $\delta_0 + \epsilon_{h,0} > 0$ such that*

$$
\begin{aligned}
S := \{\alpha(\delta, \beta, h, s, y^\delta) : 0 < (\delta + \epsilon_h) \le \delta_0 + \epsilon_{h,0} \text{ and}\\
0 < c(\delta_0 + \epsilon_{h,0}) \le \|L^{-s/2} y^\delta\|, \|y - y^\delta\| \le \delta\},
\end{aligned}
$$

is a bounded set.

Theorem 12 *Let A satisfy (14) and (17) hold. Suppose, $\hat{x} \in \mathcal{X}$, $0 < t \le \min\left\{s + \rho(s + (1-\beta)a), \frac{s-(1+\beta)a}{2}\right\}$, y^δ satisfies (2), (43) and $\alpha := \alpha(\delta, \beta, h, s, y^\delta)$ satisfies (44). Then, for $0 < \beta \le \frac{2s+a}{3a}$, $s \le a$, we have*

$$\|\hat{x} - w^{s,\delta}_{\alpha,\beta,h}\| \leq \Phi(s,a,\beta,h,t)(\delta+\epsilon_h)^{\frac{t}{(\rho+1)[(1-\beta)a+s]}},$$

where

$$\Phi(s,a,\beta,h,t) = \max\Big\{\varphi_2(s,a,\beta,h)C_{s,\beta,h,a,t}c_{s,\beta,\rho}^{\frac{s+2t}{2(\rho+1)[(1-\beta)a+s]}},$$
$$\psi(s,a,\beta,t)Ec_{s,\beta,\rho}^{\frac{t}{2(\rho+1)(\rho+1)[(1-\beta)a+s]}}\Big\},$$

$c_{s,\beta,\rho} = \frac{\|(B_{\beta,s,h}+\alpha I)^{(\rho+1)}\|}{\|y\|_{-s/2}}(c + c_{\frac{-s}{2},0})$ and $C_{s,\beta,h,a,t} = \frac{\bar{G}(\frac{-2(a+t)}{(1-\beta)a+s})}{f(\frac{s-2(a+t)}{2a})} g(\frac{s-2t}{2a})c_{t-s/2,t}E$. In particular, if $\rho = \frac{t+\beta a - s}{(1-\beta)a+s}$, then we have

$$\|\hat{x} - w^{s,\delta}_{\alpha,\beta,h}\| \leq \Phi(s,a,\beta,h,t)(\delta+\epsilon_h)^{\frac{t}{t+a}}.$$

Proof Note that, from (45), we have

$$\frac{\alpha^{\rho+1}\|y\|_{-s/2}}{\|(B_{\beta,s,h}+\alpha I)^{(\rho+1)}\|} \leq F_{s,\beta,h}(\alpha,y) \leq (c + c_{\frac{-s}{2},0})(\delta+\epsilon_h),$$

so that

$$\alpha \leq c_{s,\beta,\rho}^{1/(\rho+1)}(\delta+\epsilon_h)^{1/(\rho+1)}. \tag{47}$$

Again, by (45) and (25), we have

$$\begin{aligned}
&(c - c_{\frac{-s}{2},0})(\delta+\epsilon_h) \\
&\quad \leq \alpha^{\rho+1}\|(B_{\beta,s,h}+\alpha I)^{-(\rho+1)}L^{-s/2}y\| \\
&\quad \leq \alpha^{\rho+1}\left\|(B_{\beta,s,h}+\alpha I)^{-(\rho+1)}B_{\beta,s,h}^{\frac{a+t}{(1-\beta)a+s}}\right\|\left\|B_{\beta,s,h}^{\frac{-(a+t)}{(1-\beta)a+s}}L^{-s/2}y\right\| \\
&\quad \leq \alpha^{\frac{a+t}{(1-\beta)a+s}}\left\|B_{\beta,s,h}^{\frac{-(a+t)}{(1-\beta)a+s}}L^{-s/2}y\right\|.
\end{aligned} \tag{48}$$

Now, using Proposition 5 (3) and Proposition 2 (1), we have

$$
\begin{aligned}
\left\| B_{\beta,s,h}^{\frac{-(a+t)}{(1-\beta)a+s}} L^{-s/2} y \right\| &= \left\| B_{\beta,s,h}^{\frac{-(a+t)}{(1-\beta)a+s}} L^{-s/2} A\hat{x} \right\| \\
&\leq \bar{G}\left(\frac{-2(a+t)}{(1-\beta)a+s} \right) \| L^{-s/2} A\hat{x} \|_{a+t} \\
&\leq \frac{\bar{G}(\frac{-2(a+t)}{(1-\beta)a+s})}{f(\frac{s-2(a+t)}{2a})} \left\| A^{\frac{s-2t}{2a}} \hat{x} \right\| \\
&\leq \frac{\bar{G}(\frac{-2(a+t)}{(1-\beta)a+s})}{f(\frac{s-2(a+t)}{2a})} g\left(\frac{s-2t}{2a} \right) \|\hat{x}\|_{t-s/2} \\
&\leq \frac{\bar{G}(\frac{-2(a+t)}{(1-\beta)a+s})}{f(\frac{s-2(a+t)}{2a})} g\left(\frac{s-2t}{2a} \right) c_{t-s/2,t} \|x\|_t,
\end{aligned} \tag{49}
$$

where $c_{t-s/2,t}$ is the constant in Definition 1. Combining (48) and (49), we obtain

$$
(c - c_{\frac{-s}{2},0})\delta \leq \frac{\bar{G}(\frac{-(a+t)}{(1-\beta)a+s})}{f(\frac{s-2(a+t)}{2a})} g\left(\frac{s-2t}{2a} \right) c_{t-s/2,t} E \alpha^{\frac{a+t}{(1-\beta)a+s}},
$$

so that

$$
\delta \alpha^{\frac{-a}{(1-\beta)a+s}} \leq C_{s,\beta,h,a,t} \alpha^{\frac{t}{(1-\beta)a+s}}. \tag{50}
$$

Therefore, the result follows from Theorems 9, (47) and (50). $\square$

6 Numerical examples

Let $L : D(L) \subset \mathcal{X} \longrightarrow \mathcal{X}$ be a mapping defined by

$$
Lx := \sum_{j=1}^{\infty} j^2 \langle x, u_j \rangle u_j,
$$

where $u_j(t) = \sqrt{2}\sin(j\pi t)$ for each $j \in \mathbb{N}$ with domain of L as

$$
D(L) := \left\{ x \in L^2[0,1] : \sum_{j=1}^{\infty} j^4 |\langle x, u_j \rangle|^2 < \infty \right\}.
$$

In this case, the Hilbert scale $\{\mathcal{X}\}_s$ generated by L is given by

$$
\begin{aligned}
\mathcal{X}_s &= \left\{ x \in L^2[0,1] : \sum_{j=1}^{\infty} j^{4s} |\langle x, u_j \rangle|^2 < \infty \right\} \\
&= \left\{ x \in H^{2s}(0,1) : x^{(2l)}(0) = x^{(2l)}(1) = 0,\ l = 0, 1, \ldots, \lceil \frac{s}{2} - \frac{1}{4} \rceil \right\},
\end{aligned} \tag{51}
$$

where $\lceil p \rceil$ denotes the greatest integer less than or equal to $p, s \in \mathbb{R}$ and H^s is the usual Sobolev space. Also, one can see that $H^0 = L^2[0,1]$ and, for each $s \in \mathbb{N}$, $H_s \subset H^s$.

Now, we consider four examples to validate the theoretical results. We use finite dimensional subspaces (V_n) of $\mathcal{X}$ and $P_h(h = \frac{1}{n}) : \mathcal{X} \longrightarrow V_n$ are orthogonal projection. We choose V_n as the linear span of $\{v_1, v_2, \ldots, v_n\}$ with v_i for each $i = 1, 2, \ldots, n$ as the L^2- orthogonalized characteristic functions of the interval $[\frac{i-1}{n}, \frac{i}{n}]$. Then, since $w^{s,\delta}_{\alpha,\beta,h} \in V_n$, it is of the form $\sum_{i=1}^{n} \lambda_i v_i$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are scalars. It can be seen that $w^{s,\delta}_{\alpha,\beta,h} = \sum_{i=1}^{n} \lambda_i v_i$ is the solution of (21) if and only if $\bar{\lambda} = (\lambda_1, \cdots, \lambda_n)^T$ is the solution of

$$(M_n + \alpha B_n)\bar{\lambda} = W_n,$$

where

$$M_n = \langle A_h v_i, v_j \rangle, \quad \forall i,j = 1, 2, \ldots, n,$$
$$B_n = \langle L^s v_i, A_h^{\beta} v_j \rangle, \quad \forall i,j = 1, 2, \ldots, n,$$

and

$$W_n = (\langle y^{\delta}, v_1 \rangle, \ldots, \langle y^{\delta}, v_n \rangle)^T.$$

Here and below, $(a_1, a_2, \ldots, a_n)^T$ denote the transpose of $(a_1, a_2, \ldots, a_n)$. We have used SVD as in [35] for our computations.

Newton's method is used to solve the nonlinear equations (44) for α. The relative error $E_{\alpha,\beta} := \left(\frac{\|\hat{x} - w^{s,\delta}_{\alpha,\beta}\|}{\|w^{s,\delta}_{\alpha,\beta}\|} \right)$ and α are given in the tables for β, $\rho = \frac{t+\beta a - s}{(1-\beta)a+s}$ with $t = -\beta a + s/3$, $n = 300$ (mesh size) and noise δ.

We have used the random noise $\delta = 0.1,\ 0.01$ and 0.001 and $\epsilon_h = \epsilon_{\frac{1}{n}} = \frac{1}{n^2}$.

Example 1 ([38, Shaw]) Let

$$[Tx](s) := \int_{-\pi}^{\pi} k(s,t)x(t)dt = g(s), \quad -\pi \le s \le \pi, \tag{52}$$

where $k(s,t) = (\cos(s) + \cos(t))^2 (\frac{\sin(u)}{u})^2$, $u = \pi(\sin(s) + \sin(t))$.

We take $A := T^*T|_{N(T^*T)^{\perp}}$ and $y = T^*g$ for our computation.

The solution $\hat{x}$ is given by $\hat{x}(t) = a_1 \exp(-c_1(t - t_1)^2) + a_2 \exp(-c_2(t - t_2)^2)$.We have taken $s = a = \frac{1}{2}, d_1 = d_2 = \frac{1}{\pi^2}$ in our computation. The relative error and α values are given in Tables 1 and Table 2. The figures for exact data and noise data for

Table 1 Relative errors for fixed α

β	0	0.05	0.1	0.15	0.2
$E_{\alpha,\beta}$	$6.801195e-01$	$6.766239e-01$	$6.726160e-01$	$6.679884e-01$	$6.626066e-01$

Table 2 Relative errors under discrepancy principle

δ	β	0	0.05	0.1	0.15	0.2
	ρ	5.000000e−01	5.128205e−01	5.263158e−01	5.405405e−01	5.555556e−01
0.1	$\alpha(k)$	5.931937e−04	6.229828e−04	6.892837e−04	7.511112e−04	8.620988e−04
	$E_{\alpha,\beta}$	2.016406e−01	1.936678e−01	2.040882e−01	1.878488e−01	4.261247e−01
0.01	$\alpha(k)$	5.929687e−04	6.232274e−04	6.893498e−04	7.513056e−04	8.627321e−04
	$E_{\alpha,\beta}$	9.689599e−02	9.355126e−02	8.139111e−02	9.526484e−02	9.511034e−02
0.001	$\alpha(k)$	5.929932e−04	6.232488e−04	6.893604e−04	7.513294e−04	8.627484e−04
	$E_{\alpha,\beta}$	9.617521e−02	8.861288e−02	8.175922e−02	7.492941e−02	7.269160e−02

$\delta = 0.01$ is given in Fig. 2a, solutions with $\delta = 0.01$ and for $\beta = 0, 0.05, 0.1, 0.15$ and $\beta = 0.2$ are given in Fig. 2b–f. The sub-figure (a) contains the noise data and exact data and remaining sub-figures contain the exact solution (exact sol.) and computed solution (C.S).

Example 2 ([34, Phillips]) Let

$$\int_{-6}^{6} k(s,t)x(t)dt = g(s), \quad -6 \le u \le 6, \tag{53}$$

where $k(s,t) = \phi(s-t)$ with

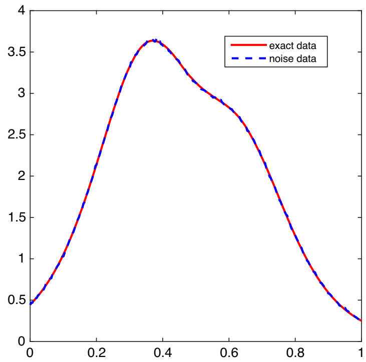
$$\phi(x) = \begin{cases} 1 + \cos(x * \pi/3), & |x| < 3, \\ 0, & |x| \ge 3. \end{cases}$$

We take $A := T^*T|_{N(T^*T)^\perp}$ and $y = T^*g$, where

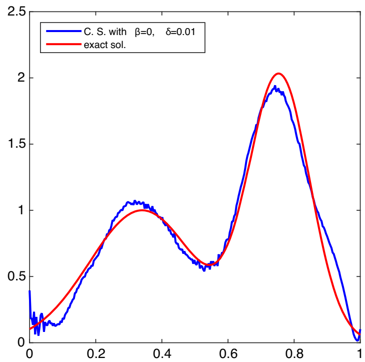
$$g(s) = (6 - |s|) * (1 + .5 * \cos(s * \pi/3)) + 9/(2 * \pi) * \sin(|s| * \pi/3),$$

for our computation. The solution $\hat{x}$ is given by $\hat{x}(t) = \phi(t)$. We have taken $s = a = \frac{1}{2}, d_1 = d_2 = \frac{1}{36}$ in our computation. The relative error and α values are given in Table 3 and Table 4. The figures for noise data and exact data for $\delta = 0.01$ is given in Fig. 3, solutions with $\delta = 0.01$ and for $\beta = 0, 0.05, 0.1, 0.15$ and $\beta = 0.2$ are given in Fig. 3b–f. The sub-figure (a) contains the noise data and exact data and remaining sub-figures contain the exact solution (exact sol.) and computed solution (C.S).

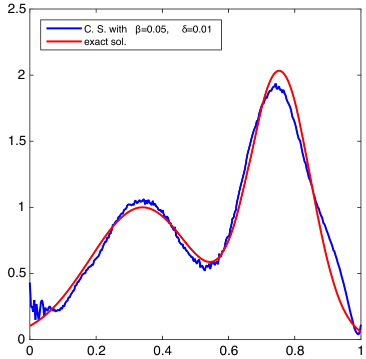
Example 3 (Non-smooth Signal) In this example we generate a square wave with sharp edges to analyse the performance of the Lavrentiev and fractional Lavrentiev method. The Lavrentiev regularization results in smoothing of sharp discontinuities where as the fractional Lavrentiev retains the sharpness in the signal thus reducing the over-smoothing effect. We have taken $s = a = \frac{1}{2}$ in our computation. The results are shown in Fig. 4b–d. The results of the proposed fractional Lavrentiev regularization model are shown in Fig. 4c, d for different β values.



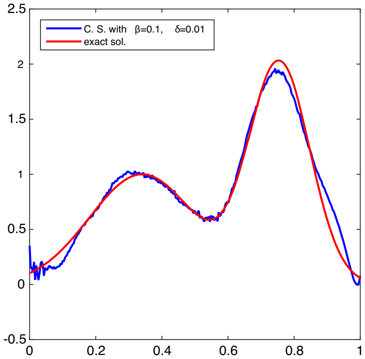
(a) Noise data and exact data for $\delta = 0.01$.



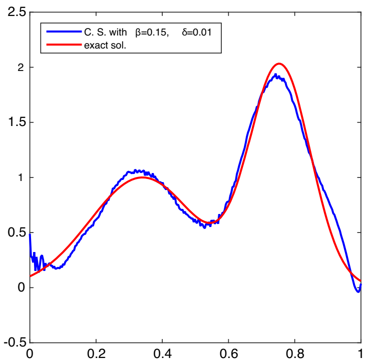
(b) Solutions with $\delta = 0.01$ and $\beta = 0$.



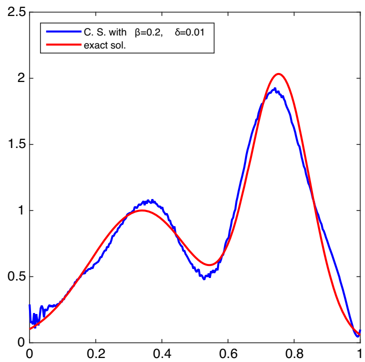
(c) Solutions with $\delta = 0.01$ and $\beta = 0.05$.



(d) Solutions with $\delta = 0.01$ and $\beta = 0.1$.



(e) Solutions with $\delta = 0.01$ and $\beta = 0.15$.



(f) Solutions with $\delta = 0.01$ and $\beta = 0.2$.

Fig. 2 Exact and evaluated solutions for different δ and β for Shawn example

Table 3 Relative errors for fixed α

β	0	0.05	0.1	0.15	0.2
$E_{\alpha,\beta}$	$6.303972e-01$	$6.253094e-01$	$6.174618e-01$	$6.046162e-01$	$5.830876e-01$

Table 4 Relative errors with discrepancy principle

δ	β ρ	0 5.000000e−01	0.05 5.263158e−01	0.1 5.405405e−01	0.15 5.555556e−01	0.2 5.555556e−01
0.1	$\alpha(k)$	1.805922e−02	2.575646e−02	3.576999e−02	4.785191e−02	6.326033e−02
	$E_{\alpha,\beta}$	3.515146e−01	2.315353e−01	3.511560e−01	3.068689e−01	2.925603e−01
0.01	$\alpha(k)$	1.807031e−02	2.576193e−02	4.825071e−02	4.823351e−02	6.338610e−02
	$E_{\alpha,\beta}$	6.661885e−02	5.778094e−02	2.703130e−02	4.506687e−02	3.913097e−02
0.001	$\alpha(k)$	1.806768e−02	2.578385e−02	3.576453e−02	4.824909e−02	6.336271e−02
	$E_{\alpha,\beta}$	2.007506e−02	1.797936e−02	2.382614e−02	2.033829e−02	2.208742e−02

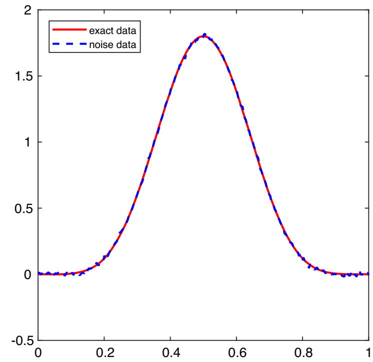
In the next example, we consider an image restoration problem:

Example 4 (Image Restoration Example) Here we show some examples to demonstrate the restoration ability of the method when applied to different images. IR Tool: a Matlab package for iterative inverse problems in [4] and Algebraic IR Tools in [13] are being used here for the numerical implementation of the model for 2D images (both gray-scale and color).

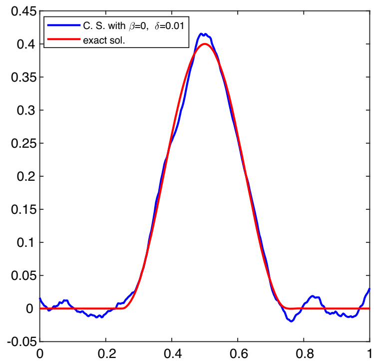
Two test images (a satellite image and a synthetic image) given along with the IR/AIR tools (package) are tested and the results are demonstrated below. The test image is synthetically corrupted by Gaussian blur with standard deviation 2 and Gaussian white noise with zero mean and noise variance 0.05. The test results are shown for standard Lavrentiev regularization and the proposed model (fractional Lavrentiev model). The standard Lavrentiev model tends to perform denoising by penalizing the image details resulting in an over-smoothed data as observed from the results. Nevertheless, the proposed model restores the images without compromising much on the details.

The original, noisy, and restored images are shown in Figs. 5 and 6 for the two different input test images. The proposed restoration process is observed to denoise the data and preserve the details as observed from the results shown for different β values. A statistical quantification has been performed using the well-known measure: Signal to Noise Ratio (SNR)[1]. The SNR of the noisy and restored versions of the test images for different β values are given in Table 5. The SNR measure being inversely proportional to the root mean square error, it increases with decrease in β value unlike the relative error.

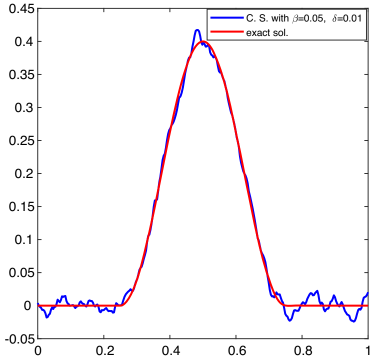
[1] SNR=$20\log_{10}\frac{\sum_{i=0}^{N}\sum_{j=0}^{M}\hat{x}(i,j)^2}{\sum_{i=0}^{N}\sum_{j=0}^{M}[\hat{x}(i,j)^2-x(i,j)]^2}$ dB, where x and $\hat{x}$ are the original and restored images, respectively.



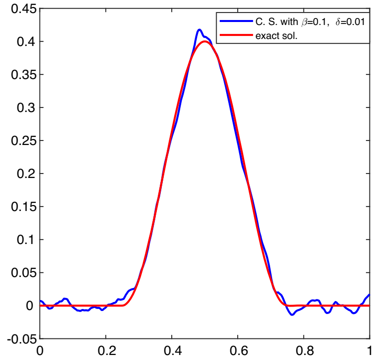
(a) Noise data and exact for $\delta = 0.01$.



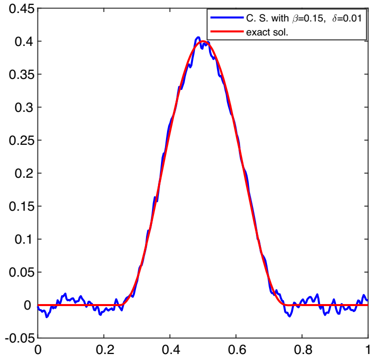
(b) Solutions with $\delta = 0.01$ and $\beta = 0$.



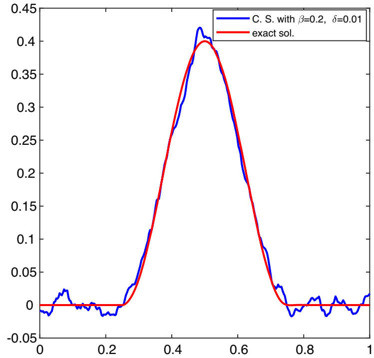
(c) Solutions with $\delta = 0.01$ and $\beta = 0.05$.



(d) Solutions with $\delta = 0.01$ and $\beta = 0.1$.

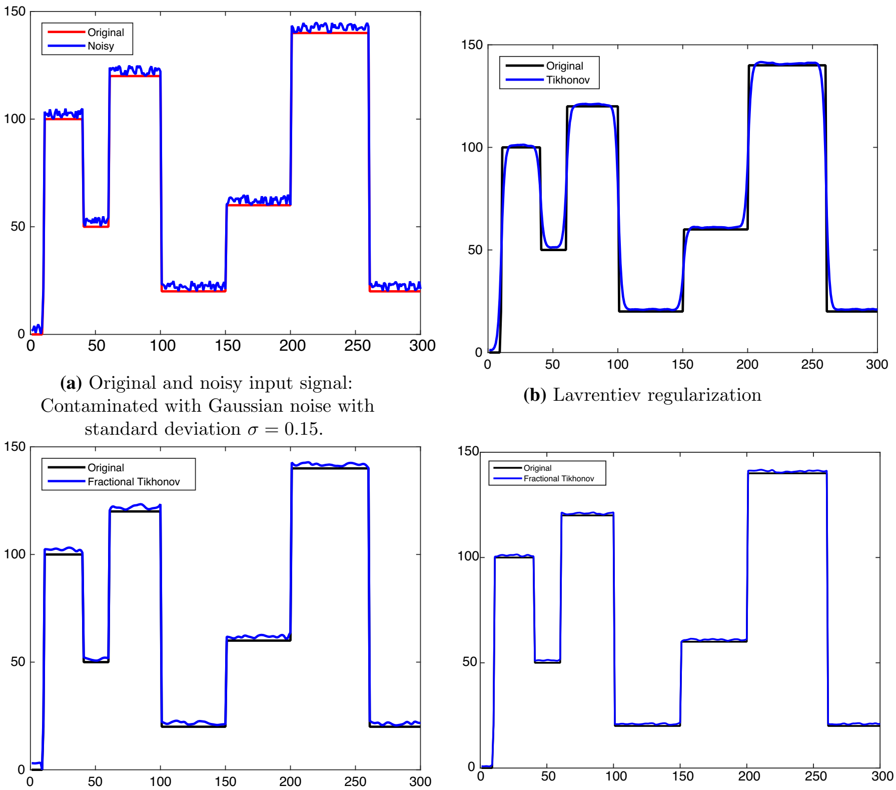


(e) Solutions with $\delta = 0.01$ and $\beta = 0.15$.



(f) Solutions with $\delta = 0.01$ and $\beta = 0.2$.

Fig. 3 Exact and evaluated solutions for different δ and β for Phillips example



(a) Original and noisy input signal: Contaminated with Gaussian noise with standard deviation $\sigma = 0.15$.

(b) Lavrentiev regularization

(c) Fractional Lavrentiev: $\beta = 0.1$

(d) Fractional Lavrentiev: $\beta = 0.2$

Fig. 4 Exact and evaluated solutions for different β

7 Conclusion

In this paper, we study the finite dimensional realization of the FLR method for approximately solving the (linear ill-posed operator) equation $A(x) = y$ in Hilbert Scales. We have also studied an a-priori and a-posteriori parameter choice strategy and obtained an optimal order error estimate under each of them. The FLR method reduces the over-smoothing in the SLR method in Hilbert space and Hilbert scales as mentioned in the introduction. The regularization saturation for the FLR method is $t = s + (1 - \beta)a/2$, whereas that of the SLR method is $t = s + a > s + (1 - \beta)a/2$. The choice of optimal value for β is still an open problem.

We have applied the methods to various well-known examples in literature and also to an image restoration problem. The magnitude of smoothing(regularization) with respect to β can be seen from the example given for image restoration problem. As the noise variance increases, the value of β also needs to be decreased in order to obtain a proper restoration. Nevertheless the blurring artifacts start appearing in the resultant data as β increases. The value of β should provide a trade-off between smoothing and deblurring as these two are two complementary requirements.

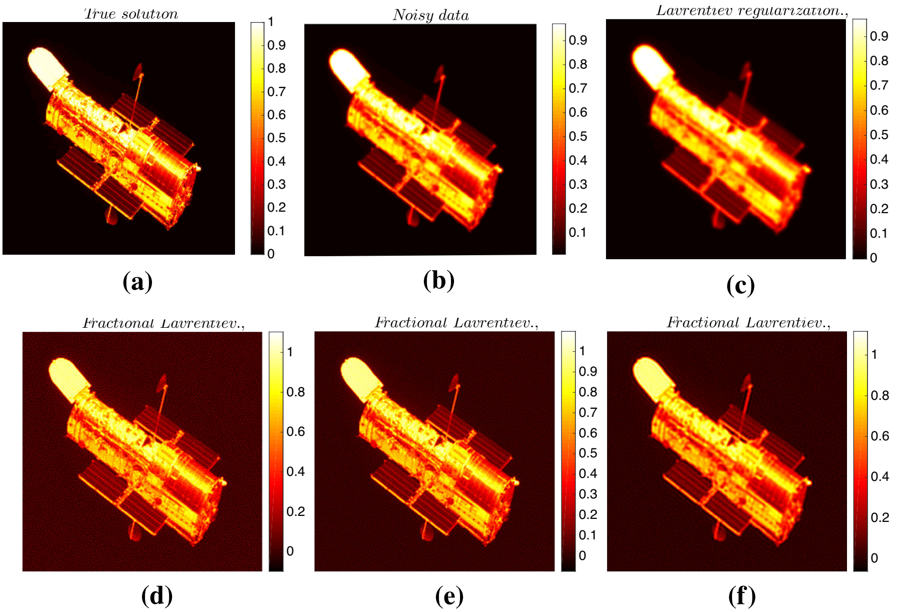


Fig. 5 **a** Original image, **b** blurred and noisy image, **c** restored using Lavrentiev regularization and **d**–**f** restored using the proposed model for $\beta = 0.1, 0.15, 0.2$, respectively

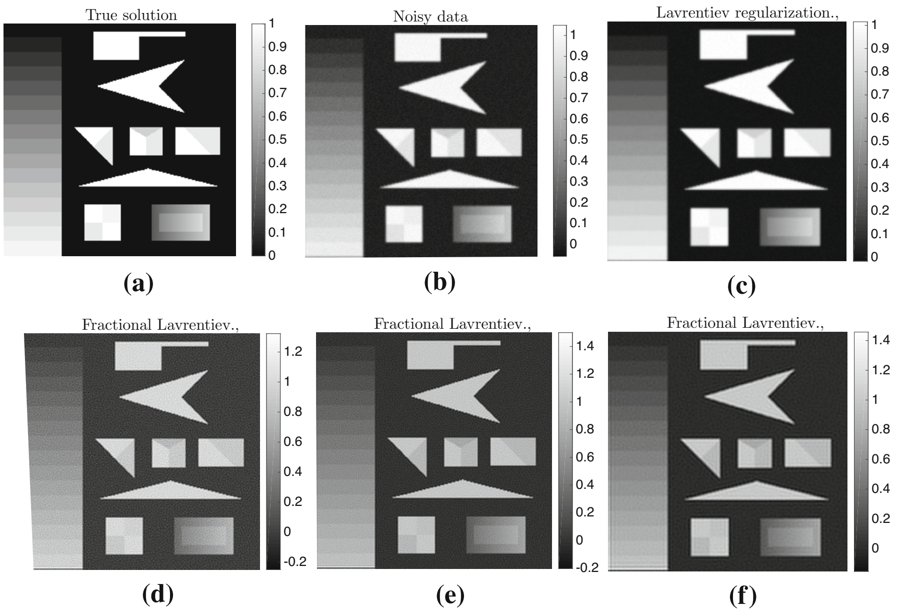


Fig. 6 **a** Original image, **b** blurred and noisy image, **c** restored using Lavrentiev regularization and **d**–**f** restored using the proposed model for $\beta = 0.1, 0.15, 0.2$, respectively

Table 5 SNR evaluated (in dB) for different β values for two different images

Image	β	Noisy & Blurred Image	Restored by the proposed
	0		4.12
	0.05		5.32
Satellite	0.1	1.92	6.41
	0.15		7.22
	0.2		8.19
	0		5.21
	0.05		6.23
Synthetic	0.1	2.32	7.22
	0.15		8.12
	0.2		8.99

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Availability of data and material The datasets used and/or analysed during the current study are available from the corresponding author on reasonable request.

Code availability Not applicable.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest. The authors alone are responsible for the content and writing of this article.

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